

# VANISHING THEOREM FOR TRANSVERSE DIRAC OPERATORS ON RIEMANNIAN FOLIATIONS

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**ABSTRACT.** We obtain a vanishing theorem for the half-kernel of a transverse  $\text{Spin}^c$  Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation twisted by a sufficiently large power of a line bundle, whose curvature vanishes along the leaves and is transversely non-degenerate at any point of the ambient manifold.

## INTRODUCTION

Let  $X$  be a compact manifold of dimension  $2n$  equipped with an almost complex structure  $J : TX \rightarrow TX$ ,  $\mathcal{E}$  a Hermitian vector bundle on  $X$ , and  $g_X$  a Riemannian metric on  $X$ . Assume that the almost complex structure  $J$  is compatible with  $g_X$ . Consider a Hermitian line bundle  $\mathcal{L}$  over  $X$  endowed with a Hermitian connection  $\nabla^{\mathcal{L}}$  such that its curvature  $R^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$  is non-degenerate. Thus,  $\omega = \frac{i}{2\pi} R^{\mathcal{L}}$  is a symplectic form on  $X$ . One can construct canonically a  $\text{Spin}^c$  Dirac operator  $D_k$  acting on

$$\Omega^{0,*}(X, \mathcal{E} \otimes \mathcal{L}^k) = \oplus_{q=0}^n \Omega^{0,q}(X, \mathcal{E} \otimes \mathcal{L}^k),$$

the direct sum of spaces of  $(0, q)$ -forms with values in  $\mathcal{E} \otimes \mathcal{L}^k$ .

Under the assumption that  $J$  is compatible with  $\omega$ , Borthwick and Uribe [2] proved that, for sufficiently large  $k$ ,

$$\text{Ker } D_k^- = 0,$$

where  $D_k^-$  denotes the restriction of  $D_k$  to  $\Omega^{0, \text{odd}}(X, \mathcal{E} \otimes \mathcal{L}^k)$ . This result generalizes the famous Kodaira vanishing theorem for the cohomology of the sheaf of sections of a holomorphic vector bundle twisted by a large power of a positive line bundle. It has interesting applications in geometric quantization (see [2] and references therein).

In [18], Ma and Marinescu gave a proof of the Borthwick-Uribe result, which uses only the Lichnerowicz formula for the  $\text{Spin}^c$  Dirac operator. They also show that, if we put

$$m = \inf_{u \in T_x^{(1,0)} X, x \in X} \frac{R_x^{\mathcal{L}}(u, \bar{u})}{|u|^2} > 0,$$

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then there exists  $C > 0$  such that, for  $k \in \mathbb{N}$ , the spectrum of  $D_k^2$  is contained in the set  $\{0\} \cup (2km - C, +\infty)$ .

Let us mention that Braverman [3, 4] generalized the Borthwick-Urbe vanishing theorem to the case when the almost complex structure  $J$  is compatible with  $g_X$ , the curvature  $R^\mathcal{L}$  is non-degenerate and  $J$ -invariant, but not necessarily compatible with  $J$ . In [19], Ma and Marinescu gave a proof of this result by the methods of [18].

Our main purpose is to obtain an analogue of the Borthwick-Urbe vanishing theorem for a transverse  $\text{Spin}^c$  Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation. Our considerations are based on the approach of Ma-Marinescu [18]. So we also state a Lichnerowicz type formula for a transverse Dirac operator on a compact foliated manifold  $(M, \mathcal{F})$ , which, as we strongly believe, will be of independent interest.

The transverse Dirac operators for Riemannian foliations were introduced in [7]. This paper mainly deals with the transverse Dirac operators acting on basic sections (see also [8, 9, 11, 12, 13] and references therein). The index theory of transverse Dirac operators was studied in [6]. Finally, spectral triples defined by transverse Dirac-type operators on Riemannian foliations and related noncommutative geometry were considered in [15, 16, 17].

The paper is organized as follows. In Section 1, we introduce transverse Dirac operators and formulate our main results, the vanishing theorem for a transverse  $\text{Spin}^c$  Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation, Theorem 3, and the Lichnerowicz formula for transverse Dirac operators, Theorem 4. The proof of the vanishing theorem is given in Section 2. Sections 3 and 4 contain the proof of the Lichnerowicz formula and related results.

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## 1. PRELIMINARIES AND MAIN RESULTS

**1.1. Transverse Dirac operators.** Let  $M$  be a compact manifold equipped with a Riemannian foliation  $\mathcal{F}$  of even codimension  $q$  and  $g_M$  a bundle-like metric on  $M$ . Let  $T_x^H M = T_x \mathcal{F}^\perp$ .  $T^H M$  is a smooth vector subbundle of  $TM$  such that

$$(1) \quad TM = T^H M \oplus T\mathcal{F}.$$

There is a natural isomorphism  $T^H M \cong Q = TM/T\mathcal{F}$ . Denote by  $P_H$  (resp.  $P_F$ ) the orthogonal projection operator of  $TM$  on  $T^H M$  (resp.  $T\mathcal{F}$ ) associated with the decomposition (1).

The Riemannian metric  $g_M$  gives rise to a metric connection  $\nabla$  in  $T^H M$  (called the transverse Levi-Civita connection), which is defined as follows.

Denote by  $\nabla^L$  the Levi-Civita connection defined by  $g_M$ . Then we have

$$(2) \quad \begin{aligned} \nabla_X N &= P_H[X, N], \quad X \in C^\infty(M, T\mathcal{F}), \quad N \in C^\infty(M, T^H M) \\ \nabla_X N &= P_H \nabla_X^L N, \quad X \in C^\infty(M, T^H M), \quad N \in C^\infty(M, T^H M). \end{aligned}$$

It turns out that  $\nabla$  depends only on the transverse part of the metric  $g_M$  and preserves the inner product of  $T^H M$ .

For any  $x \in M$ , denote by  $Cl(Q_x)$  the Clifford algebra of  $Q_x$ . Recall that, relative to an orthonormal basis  $\{f_1, f_2, \dots, f_q\}$  of  $Q_x$ ,  $Cl(Q_x)$  is the complex algebra generated by 1 and  $f_1, f_2, \dots, f_q$  with relations

$$f_\alpha f_\beta + f_\beta f_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \dots, q.$$

The transverse Clifford bundle  $Cl(Q)$  is the  $\mathbb{Z}_2$ -graded vector bundle over  $M$  whose fiber at  $x \in M$  is  $Cl(Q_x)$ . This bundle is associated to the principal  $SO(q)$ -bundle  $O(Q)$  of oriented orthonormal frames in  $Q$ ,  $Cl(Q) = O(Q) \times_{O(q)} Cl(\mathbb{R}^q)$ . Therefore, the transverse Levi-Civita connection  $\nabla$  induces a natural leafwise flat connection  $\nabla^{Cl(Q)}$  on  $Cl(Q)$  which is compatible with the multiplication and preserves the  $\mathbb{Z}_2$ -grading on  $Cl(Q)$ . If  $\{f_1, f_2, \dots, f_q\}$  is a local orthonormal frame in  $T^H M$ , and  $\omega_{\alpha\beta}^\gamma$  is the coefficients of the connection  $\nabla$ :  $\nabla_{f_\alpha} f_\beta = \sum_\gamma \omega_{\alpha\beta}^\gamma f_\gamma$ , then

$$(3) \quad \nabla_{f_\alpha}^{Cl(Q)} = f_\alpha + \frac{1}{4} \sum_{\gamma=1}^q \omega_{\alpha\beta}^\gamma c(f_\beta) c(f_\gamma),$$

where  $c(a)$  denotes the action of an element  $a \in Cl(Q)$  on  $C^\infty(M, Cl(Q))$  by pointwise left multiplication.

A transverse Clifford module is a complex vector bundle  $\mathcal{E}$  on  $M$  endowed with an action of the bundle  $Cl(Q)$ . We will denote the action of  $a \in C^\infty(M, Cl(Q))$  on  $s \in C^\infty(M, \mathcal{E})$  as  $c(a)s \in C^\infty(M, \mathcal{E})$ .

A transverse Clifford module  $\mathcal{E}$  is called self-adjoint if it endowed with a leafwise flat Hermitian metric such that the operator  $c(f) : \mathcal{E}_x \rightarrow \mathcal{E}_x$  is skew-adjoint for any  $x \in M$  and  $f \in Q_x$ .

Any transverse Clifford module  $\mathcal{E}$  carries a natural  $\mathbb{Z}_2$ -grading  $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$  (see, for instance, [1]).

A connection  $\nabla^\mathcal{E}$  on a transverse Clifford module  $\mathcal{E}$  is called a Clifford connection if it is compatible with the Clifford action, that is, for any  $f \in C^\infty(M, T^H M)$  and  $a \in C^\infty(M, Cl(Q))$ ,

$$[\nabla_f^\mathcal{E}, c(a)] = c(\nabla_f^{Cl(Q)} a).$$

**Example 1.** Assume that  $\mathcal{F}$  is transversely oriented and the normal bundle  $Q$  is spin. Thus the  $SO(q)$  bundle  $O(Q)$  of oriented orthonormal frames in  $Q$  can be lifted to a  $Spin(q)$  bundle  $O'(Q)$  so that the projection  $O'(Q) \rightarrow O(Q)$  induces the covering projection  $Spin(q) \rightarrow SO(q)$  on each fiber.

Let  $F(Q), F_+(Q), F_-(Q)$  be the bundles of spinors

$$F(Q) = O'(Q) \times_{Spin(q)} S, \quad F_\pm(Q) = O'(Q) \times_{Spin(q)} S_\pm.$$

Since  $\dim Q = q$  is even  $\text{End } F(Q)$  is as a bundle of algebras over  $M$  isomorphic to the Clifford bundle  $Cl(Q)$ . So  $F(Q)$  is a self-adjoint transverse Clifford module. The transverse Levi-Civita connection  $\nabla$  lifts to a leafwise flat Clifford connection  $\nabla^{F(Q)}$  on  $F(Q)$ .

More generally, one can take a Hermitian vector bundle  $\mathcal{W}$  equipped with a leafwise flat Hermitian connection  $\nabla^{\mathcal{W}}$ . Then  $F(Q) \otimes \mathcal{W}$  is a transverse Clifford module: the action of  $a \in C^\infty(M, Cl(Q))$  on  $C^\infty(M, F(Q) \otimes \mathcal{W})$  is given by  $c(a) \otimes 1$  ( $c(a)$  denotes the action of  $a$  on  $C^\infty(M, F(Q))$ ). The product connection  $\nabla^{F(Q) \otimes \mathcal{W}} = \nabla^{F(Q)} \otimes 1 + 1 \otimes \nabla^{\mathcal{W}}$  on  $F(Q) \otimes \mathcal{W}$  is a Clifford connection.

**Example 2.** Another example of a self-adjoint transverse Clifford module associated with a transverse almost complex structure on  $(M, \mathcal{F})$ , a transverse Clifford module  $\Lambda^{0,*}$ , is described in Section 1.2.

Let  $\mathcal{E}$  be a self-adjoint transverse Clifford module equipped with a leafwise flat Clifford connection  $\nabla^{\mathcal{E}}$ . We will identify the bundle  $Q$  and  $Q^*$  by means of the metric  $g_M$  and define the operator  $D'_\mathcal{E}$  acting on the sections of  $\mathcal{E}$  as the composition

$$C^\infty(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(M, Q^* \otimes \mathcal{E}) = C^\infty(M, Q \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E}).$$

This operator is odd with respect to the natural  $\mathbb{Z}_2$ -grading on  $\mathcal{E}$ . If  $f_1, \dots, f_q$  is a local orthonormal frame for  $T^H M$ , then

$$D'_\mathcal{E} = \sum_{\alpha=1}^q c(f_\alpha) \nabla_{f_\alpha}^{\mathcal{E}}.$$

Let  $\tau \in C^\infty(M, T^H M)$  be the mean curvature vector field of  $\mathcal{F}$ . If  $e_1, e_2, \dots, e_p$  is a local orthonormal frame in  $T\mathcal{F}$ , then

$$\tau = \sum_{i=1}^p P_H(\nabla_{e_i}^L e_i).$$

The transverse Dirac operator  $D_\mathcal{E}$  is defined as

$$D_\mathcal{E} = D'_\mathcal{E} - \frac{1}{2}c(\tau) = \sum_{\alpha=1}^q c(f_\alpha) \left( \nabla_{f_\alpha}^{\mathcal{E}} - \frac{1}{2}g_M(\tau, f_\alpha) \right).$$

Denote by  $(\cdot, \cdot)_x$  the inner product in the fiber  $\mathcal{E}_x$  over  $x \in M$ . Then the inner product in  $L^2(M, \mathcal{E})$  is given by the formula

$$(s_1, s_2) = \int_M (s_1(x), s_2(x))_x \omega_M, \quad s_1, s_2 \in L^2(M, \mathcal{E}),$$

where  $\omega_M = \sqrt{\det g} dx$  denotes the Riemannian volume form on  $M$ . As shown in [17], the transverse Dirac operator  $D_\mathcal{E}$  is formally self-adjoint in  $L^2(M, \mathcal{E})$ .

**1.2. The vanishing theorem.** As above, let  $M$  be a compact manifold equipped with a Riemannian foliation  $\mathcal{F}$  of even codimension  $q$ ,  $g_M$  a bundle-like metric on  $M$ . Consider a Hermitian line bundle  $\mathcal{L}$  equipped with a leafwise flat Hermitian connection  $\nabla^{\mathcal{L}}$ .

The curvature of  $\nabla^{\mathcal{L}}$  is an imaginary valued 2-form  $R^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$  on  $M$ . Since  $\nabla^{\mathcal{L}}$  is leafwise flat,  $R^{\mathcal{L}}$  vanishes on  $T\mathcal{F}$ , and, therefore, defines a 2-form  $R^{\mathcal{L}}$  on  $Q$ . If this form is non-degenerate, then it is a symplectic form on  $Q$ .

Let  $J : Q \rightarrow Q$  be an almost complex structure, which is compatible with  $g_Q$  and  $R^{\mathcal{L}}$ . The almost complex structure  $J$  defines canonically an orientation of  $Q$  and induces a splitting  $Q \otimes \mathbb{C} = Q^{(1,0)} \oplus Q^{(0,1)}$ , where  $Q^{(1,0)}$  and  $Q^{(0,1)}$  are the eigenbundles of  $J$  corresponding to the eigenvalues  $i$  and  $-i$  respectively. We also have the corresponding decomposition of the complexified conormal bundle  $Q^* \otimes \mathbb{C} = Q^{(1,0)*} \oplus Q^{(0,1)*}$  and the decomposition of the exterior algebra bundles  $\Lambda(Q^* \otimes \mathbb{C}) = \bigoplus_{p,q} \Lambda^{p,q}(Q^* \otimes \mathbb{C})$ , where  $\Lambda^{p,q}(Q^* \otimes \mathbb{C}) = \Lambda^p Q^{(1,0)*} \otimes \Lambda^q Q^{(0,1)*}$ . The transverse Levi-Civita connection  $\nabla$  can be written as

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)} + A,$$

where  $\nabla^{(1,0)}$  and  $\nabla^{(0,1)}$  are the canonical Hermitian connections on  $Q^{(1,0)}$  and  $Q^{(0,1)}$  respectively and  $A \in C^\infty(T^*M \otimes \text{End}(Q))$ , which satisfies  $JA = -AJ$ .

Consider a self-adjoint transverse Clifford module

$$\Lambda^{0,*} = \Lambda^{\text{even}} Q^{(0,1)*} \oplus \Lambda^{\text{odd}} Q^{(0,1)*}.$$

The action of any  $f \in Q$  with decomposition  $f = f_{1,0} + f_{0,1} \in Q^{(1,0)} \oplus Q^{(0,1)}$  on  $\Lambda^{0,*}$  is defined as

$$c(f) = \sqrt{2}(\varepsilon_{f_{1,0}^*} - i_{f_{0,1}}),$$

where  $\varepsilon_{f_{1,0}^*}$  denotes the exterior product by the covector  $f_{1,0}^* \in Q_x^*$  dual to  $f_{1,0}$ ,  $i_{f_{0,1}}$  the interior product by  $f_{0,1}$ . This module has a natural leafwise flat Clifford connection  $\nabla^{\Lambda^{0,*}}$ . The associated transverse Dirac operator  $D_{\Lambda^{0,*}}$  can be called the transverse  $\text{Spin}^c$  Dirac operator.

One can also consider a Hermitian vector bundle  $\mathcal{W}$  equipped with a leafwise flat Hermitian connection  $\nabla^{\mathcal{W}}$ . Then one gets the twisted transverse Clifford module  $\mathcal{E} = \Lambda^{0,*} \otimes \mathcal{W}$  equipped with a product leafwise flat Hermitian connection  $\nabla^{\mathcal{E}}$  and the associated transverse  $\text{Spin}^c$  Dirac operator  $D_{\Lambda^{0,*} \otimes \mathcal{W}}$ .

Consider the transverse  $\text{Spin}^c$  Dirac operator

$$D_k = D_{\Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k} : C^\infty(M, \Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k) \rightarrow C^\infty(M, \Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k).$$

Let  $D_k^-$  denote the restriction of  $D_k$  to the space  $C^\infty(M, \Lambda^{\text{odd}} Q^{(0,1)*} \otimes \mathcal{W} \otimes \mathcal{L}^k)$ . Put

$$m = \inf_{u \in Q_x^{(1,0)}, x \in M} \frac{R_x^{\mathcal{L}}(u, \bar{u})}{|u|^2} > 0.$$

**Theorem 3.** *There exists  $C > 0$  such that for  $k \in \mathbb{N}$ , the spectrum of  $D_k^2$  is contained in the set  $\{0\} \cup (2km - C, +\infty)$ . For sufficiently large  $k$*

$$\text{Ker } D_k^- = 0.$$

The proof of this theorem will be given in Section 2.

**1.3. The Lichnerowicz formula.** In this Section, we will formulate the Lichnerowicz formula for a transverse Dirac operator, which will play a crucial role in the proof of Theorem 3.

Denote by  $\mathcal{R}$  the integrability tensor (or curvature) of  $T^H M$ . It is an element of  $C^\infty(M, \Lambda^2 T^H M^* \otimes T\mathcal{F})$  given by

$$\mathcal{R}_x(f_1, f_2) = -P_F[\tilde{f}_1, \tilde{f}_2](x), \quad x \in M, \quad f_1, f_2 \in T_x^H M,$$

where, for any  $f \in T_x^H M$ ,  $\tilde{f} \in C^\infty(M, T^H M)$  denotes any vector field, which coincides with  $f$  at  $x$ .

Since the Levi-Civita connection  $\nabla^L$  is torsion-free, for any  $f_1, f_2 \in C^\infty(M, T^H M)$ , we have

$$(4) \quad \nabla_{f_1} f_2 - \nabla_{f_2} f_1 - [f_1, f_2] = \mathcal{R}(f_1, f_2).$$

Let  $R$  be the curvature of the transverse Levi-Civita connection  $\nabla$ . By definition,  $R$  is a section of  $\Lambda^2 T^* M \otimes \text{End}(T^H M)$  given by the formula

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}, \quad X, Y \in C^\infty(M, TM).$$

If  $(B, g_B)$  is a local model for the foliation and  $R^B$  is the curvature of  $g_B$ , then, for any  $f_1, f_2, f_3 \in TB$  with the corresponding horizontal lifts  $f_1^H, f_2^H, f_3^H \in T^H M$ , we have

$$R(f_1^H, f_2^H) f_3^H = [R^B(f_1, f_2) f_3]^H + P_H([\mathcal{R}(f_1^H, f_2^H), f_3^H]).$$

Denote by  $R^\mathcal{E}$  the curvature of the Clifford connection  $\nabla^\mathcal{E}$ . By definition,  $R^\mathcal{E}$  is a section of  $\Lambda^2 T_H^* M \otimes \text{End}(\mathcal{E})$  given by the formula

$$R^\mathcal{E}(f_1, f_2) = \nabla_{f_1}^\mathcal{E} \nabla_{f_2}^\mathcal{E} - \nabla_{f_2}^\mathcal{E} \nabla_{f_1}^\mathcal{E} - \nabla_{[f_1, f_2]}^\mathcal{E}.$$

It can be written as

$$R^\mathcal{E} = c(R) + R^{\mathcal{E}/S},$$

where  $c(R) \in C^\infty(M, \Lambda^2 T_H^* M \otimes Cl(Q))$  is determined by the curvature  $R$  of  $\nabla$ : If  $\{f_1, f_2, \dots, f_q\}$  is a local orthonormal frame in  $T^H M$ , then

$$c(R)(f_1, f_2) = \frac{1}{4} \sum_{\alpha, \beta} (R(f_1, f_2) f_\alpha, f_\beta) c(f_\alpha) c(f_\beta),$$

and  $R^{\mathcal{E}/S} \in C^\infty(M, \Lambda^2 T_H^* M \otimes \text{End}_{Cl(Q)}(\mathcal{E}))$  is the twisting curvature of  $\mathcal{E}$ .

Denote by  $(\nabla_X^\mathcal{E})^*$  the formal adjoint of the operator  $\nabla_X^\mathcal{E}$  with  $X \in C^\infty(M, T^H M)$  in  $L^2(M, \mathcal{E})$ . Observe the following formula:

$$(5) \quad (\nabla_X^\mathcal{E})^* = -\nabla_X^\mathcal{E} - \text{div } X.$$

where  $\operatorname{div} X \in C^\infty(M)$  denotes the divergence of  $X$ . If  $e_1, e_2, \dots, e_p$  is a local orthonormal frame in  $T\mathcal{F}$  and  $f_1, \dots, f_q$  is a local orthonormal basis of  $T^H M$ , then

$$\operatorname{div} X = \sum_{k=1}^p g_M(e_k, \nabla_{e_k} X) + \sum_{\beta=1}^q g_M(f_\beta, \nabla_{f_\beta} X).$$

In particular, it is easy to see that

$$\operatorname{div} f_\alpha = -g_M(\tau + \sum_{\beta=1}^q \nabla_{f_\beta} f_\beta, f_\alpha).$$

Let  $f_1, \dots, f_q$  be a local orthonormal basis of  $T^H M$ . Define the transverse scalar curvature  $K$  as

$$K = \sum_{\alpha, \beta} g(R(f_\alpha, f_\beta) f_\alpha, f_\beta).$$

**Theorem 4.** *The following formula holds:*

$$\begin{aligned} (D\mathcal{E})^2 &= \sum_{\alpha=1}^q (\nabla_{f_\alpha}^\mathcal{E})^* \nabla_{f_\alpha}^\mathcal{E} - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha) c(\nabla_{f_\alpha} \tau) - \frac{1}{4} \|\tau\|^2 \\ &\quad + \frac{K}{4} + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) [R^{\mathcal{E}/S}(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)}], \end{aligned}$$

where  $f_1, \dots, f_q$  is a local orthonormal basis of  $T^H M$ .

The proof of this theorem and related results will be given in Section 3.

## 2. PROOF OF THE VANISHING THEOREM

The purpose of this Section is to give the proof of Theorem 3. This proof will make an essential use of Theorem 4, whose proof will be given later, in Section 3. First, we introduce some notation.

For any  $x \in M$ , define the skew-symmetric linear map  $K_x : Q_x \rightarrow Q_x$  by the formula

$$iR^\mathcal{L}(v, w) = g_Q(v, K_x w), \quad v, w \in Q_x.$$

The eigenvalues of  $K_x$  are purely imaginary:  $\pm i\mu_j(x)$ ,  $j = 1, 2, \dots, l$  with  $\mu_j(x) > 0$ . Define

$$\lambda(x) = \operatorname{Tr}^+ K_x = \mu_1(x) + \dots + \mu_l(x), \quad m(x) = \min_j \mu_j(x).$$

Observe that

$$m = \min_{x \in M} m(x).$$

Denote

$$c(\mathcal{R}) = \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) \nabla_{\mathcal{R}(f_\alpha, f_\beta)}$$

and

$$c(R^{\mathcal{L}}) = \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) R^{\mathcal{L}}(f_\alpha, f_\beta).$$

We start the proof of Theorem 3 with the following lemma, which provides a lower estimate of the transverse metric Laplacian (cf. [18, Corollary 2.4] and reference therein).

**Lemma 5.** *Let  $\mathcal{V}$  be a Hermitian vector bundle over  $M$  equipped with a leafwise flat unitary connection  $\nabla^{\mathcal{V}}$ , and  $\mathcal{L}$  a Hermitian line bundle equipped with a leafwise flat Hermitian connection  $\nabla^{\mathcal{L}}$ . There exists  $C > 0$  such that for any  $k \in \mathbb{N}$  the transverse metric Laplacian*

$$\Delta^{\mathcal{V} \otimes \mathcal{L}^k} = \sum_{\alpha=1}^q (\nabla_{f_\alpha}^{\mathcal{V} \otimes \mathcal{L}^k})^* \nabla_{f_\alpha}^{\mathcal{V} \otimes \mathcal{L}^k}$$

satisfies

$$((\Delta^{\mathcal{V} \otimes \mathcal{L}^k} - c(\mathcal{R}))u, u) \geq k(\lambda u, u) - C\|u\|^2$$

for any  $u \in C^\infty(M, \mathcal{V} \otimes \mathcal{L}^k)$ .

*Proof.* Consider the twisted transverse Clifford module  $\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k$  and the associated twisted transverse  $\text{Spin}^c$  Dirac operator  $D_{\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k}$ . By Theorem 4, we have

$$\begin{aligned} D_{\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k}^2 &= \Delta^{\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k} - \frac{1}{2} \sum_{\alpha=1}^q (c(f_\alpha) c(\nabla_{f_\alpha} \tau) \otimes 1) - \frac{1}{4} \|\tau\|^2 + \frac{K}{4} \\ &\quad + c(R^{\mathcal{V}}) - c(\mathcal{R}) + kc(R^{\mathcal{L}}), \end{aligned}$$

where  $f_1, \dots, f_q$  is a local orthonormal basis of  $T^H M$ . From (3), we have

$$\|\nabla^{\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k} u\|^2 = \|\nabla^{\mathcal{V} \otimes \mathcal{L}^k} u\|^2 + \frac{1}{16} \left\| \sum_{\gamma} \omega_{\alpha\beta}^\gamma c(f_\beta) c(f_\gamma) u \right\|^2.$$

It can be shown (see, for instance, [3, Lemma 7.10]) that, for any  $u \in (\mathcal{V} \otimes \mathcal{L}^k)_x \subset (\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k)_x$

$$(6) \quad c(R^{\mathcal{L}})u = -\lambda u.$$

Using (6), we get, for any  $u \in C^\infty(M, \mathcal{V} \otimes \mathcal{L}^k) \subset C^\infty(M, \Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k)$ ,

$$0 \leq \|D_{\Lambda^{0,*} \otimes \mathcal{V} \otimes \mathcal{L}^k} u\|^2 \leq \|\nabla^{\mathcal{V} \otimes \mathcal{L}^k} u\|^2 + C\|u\|^2 - (c(\mathcal{R})u, u) - k(\lambda u, u),$$

that completes the proof.  $\square$

By Theorem 4, we have

$$\begin{aligned} D_k^2 &= \Delta^{\Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k} - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha) c(\nabla_{f_\alpha} \tau) - \frac{1}{4} \|\tau\|^2 + \frac{K}{4} \\ &\quad + c(R^{\mathcal{W}}) - c(\mathcal{R}) + kc(R^{\mathcal{L}}), \end{aligned}$$



where  $f_1, \dots, f_q$  is a local orthonormal basis of  $T^H M$ . Therefore, for any  $u \in C^\infty(M, \Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k)$ , we have

$$\|D_k u\|^2 \geq (\Delta^{\Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k} u, u) - (c(\mathcal{R})u, u) + k(c(R^\mathcal{L})u, u) - C\|u\|^2.$$

By Lemma 5, it follows that

$$((\Delta^{\Lambda^{0,*} \otimes \mathcal{W} \otimes \mathcal{L}^k} - c(\mathcal{R}))u, u) \geq k(\lambda u, u) - C\|u\|^2.$$

So we see that

$$\|D_k u\|^2 \geq k(\lambda u, u) + k(c(R^\mathcal{L})u, u) - C\|u\|^2.$$

Finally, by [3, Proposition 7.5], we have

$$(c(R^\mathcal{L})u, u)_x \geq -(\lambda(x) - 2m(x))\|u\|_x^2, \quad u \in (\Lambda^{odd} Q^{(0,1)*} \otimes \mathcal{W} \otimes \mathcal{L}^k)_x.$$

Therefore, for  $u \in C^\infty(M, \Lambda^{odd} Q^{(0,1)*} \otimes \mathcal{W} \otimes \mathcal{L}^k)$ , we get

$$\|D_k u\|^2 \geq 2k(mu, u) - C\|u\|^2,$$

that immediately completes the proof of Theorem 3.

### 3. PROOF OF THE LICHNEROWICZ FORMULA

In this Section, we derive the Lichnerowicz formula for a transverse Dirac operator given in Theorem 4. We start with a computation of  $(D'_\mathcal{E})^2$ :

$$\begin{aligned} (D'_\mathcal{E})^2 &= \frac{1}{2} \left[ \left( \sum_{\alpha=1}^q c(f_\alpha) \nabla_{f_\alpha}^\mathcal{E} \right) \left( \sum_{\beta=1}^q c(f_\beta) \nabla_{f_\beta}^\mathcal{E} \right) \right. \\ &\quad \left. + \left( \sum_{\beta=1}^q c(f_\beta) \nabla_{f_\beta}^\mathcal{E} \right) \left( \sum_{\alpha=1}^q c(f_\alpha) \nabla_{f_\alpha}^\mathcal{E} \right) \right] \\ &= \frac{1}{2} \sum_{\alpha, \beta} (c(f_\alpha) c(f_\beta) + c(f_\beta) c(f_\alpha)) \nabla_{f_\alpha}^\mathcal{E} \nabla_{f_\beta}^\mathcal{E} \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) (\nabla_{f_\beta}^\mathcal{E} \nabla_{f_\alpha}^\mathcal{E} - \nabla_{f_\alpha}^\mathcal{E} \nabla_{f_\beta}^\mathcal{E}) \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} [c(f_\alpha) c(\nabla_{f_\alpha} f_\beta) \nabla_{f_\beta}^\mathcal{E} + c(f_\beta) c(\nabla_{f_\beta} f_\alpha) \nabla_{f_\alpha}^\mathcal{E}]. \end{aligned}$$

For the first term, we get

$$\frac{1}{2} \sum_{\alpha, \beta} (c(f_\alpha) c(f_\beta) + c(f_\beta) c(f_\alpha)) \nabla_{f_\alpha}^\mathcal{E} \nabla_{f_\beta}^\mathcal{E} = - \sum_{\alpha} (\nabla_{f_\alpha}^\mathcal{E})^2.$$

For the second term, we get

$$\frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) (\nabla_{f_\beta}^\mathcal{E} \nabla_{f_\alpha}^\mathcal{E} - \nabla_{f_\alpha}^\mathcal{E} \nabla_{f_\beta}^\mathcal{E})$$

$$= \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) R^\mathcal{E}(f_\beta, f_\alpha) + \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) \nabla_{[f_\beta, f_\alpha]}^\mathcal{E}.$$

Let  $\nabla_{f_\alpha} f_\beta = \sum_\gamma a_{\alpha\beta}^\gamma f_\gamma$ . Since  $\nabla$  is compatible with the metric, we have  $a_{\alpha\beta}^\gamma = -a_{\alpha\gamma}^\beta$ . Thus we get

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha, \beta} [c(f_\alpha) c(\nabla_{f_\alpha} f_\beta) \nabla_{f_\beta}^\mathcal{E} + c(f_\beta) c(\nabla_{f_\beta} f_\alpha) \nabla_{f_\alpha}^\mathcal{E}] \\ &= \frac{1}{2} \sum_{\alpha, \beta, \gamma} [a_{\alpha\beta}^\gamma c(f_\alpha) c(f_\gamma) \nabla_{f_\beta}^\mathcal{E} + a_{\alpha\beta}^\gamma c(f_\beta) c(f_\gamma) \nabla_{f_\alpha}^\mathcal{E}] \\ &= -\frac{1}{2} \sum_{\alpha, \beta, \gamma} [a_{\alpha\gamma}^\beta c(f_\alpha) c(f_\gamma) \nabla_{f_\beta}^\mathcal{E} + a_{\beta\gamma}^\alpha c(f_\beta) c(f_\gamma) \nabla_{f_\alpha}^\mathcal{E}] \\ &= -\frac{1}{2} \sum_{\alpha, \gamma} [c(f_\alpha) c(f_\gamma) \nabla_{\nabla_{f_\alpha} f_\gamma}^\mathcal{E} + \sum_{\beta, \gamma} c(f_\beta) c(f_\gamma) \nabla_{\nabla_{f_\alpha} f_\gamma}^\mathcal{E}] \\ &= -\sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E}. \end{aligned}$$

From the last three identities, we get

$$\begin{aligned} (D'_\mathcal{E})^2 &= -\sum_{\alpha} (\nabla_{f_\alpha}^\mathcal{E})^2 + \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) R^\mathcal{E}(f_\beta, f_\alpha) \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) \nabla_{[f_\beta, f_\alpha]}^\mathcal{E} - \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E}. \end{aligned}$$

Consider the last two terms in this identity. Using (4), we get

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) \nabla_{[f_\beta, f_\alpha]}^\mathcal{E} - \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E} \\ &= \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) (\nabla_{\nabla_{f_\beta} f_\alpha}^\mathcal{E} - \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E} - \nabla_{\mathcal{R}(f_\beta, f_\alpha)}^\mathcal{E}) \\ &\quad - \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E} \\ &= -\frac{1}{2} \sum_{\alpha, \beta} (c(f_\alpha) c(f_\beta) + c(f_\beta) c(f_\alpha)) \nabla_{\nabla_{f_\alpha} f_\beta}^\mathcal{E} \\ &\quad - \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) \nabla_{\mathcal{R}(f_\beta, f_\alpha)}^\mathcal{E} \\ &= \sum_{\alpha} \nabla_{\nabla_{f_\alpha} f_\alpha}^\mathcal{E} - \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta) c(f_\alpha) \nabla_{\mathcal{R}(f_\beta, f_\alpha)}^\mathcal{E}. \end{aligned}$$

By (5), we also get

$$(7) \quad \sum_{\alpha=1}^q (\nabla_{f_\alpha}^\mathcal{E})^* \nabla_{f_\alpha}^\mathcal{E} = - \sum_{\alpha=1}^q (\nabla_{f_\alpha}^\mathcal{E})^2 + \nabla_\tau^\mathcal{E} + \nabla_{\sum_\alpha \nabla_{f_\alpha} f_\alpha}^\mathcal{E}.$$

Thus, we arrive at the formula

$$(8) \quad (D'_\mathcal{E})^2 = \sum_{\alpha=1}^q (\nabla_{f_\alpha}^\mathcal{E})^* \nabla_{f_\alpha}^\mathcal{E} - \nabla_\tau^\mathcal{E} + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) [R^\mathcal{E}(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)}^\mathcal{E}].$$

Now we turn to  $(D_\mathcal{E})^2$ :

$$\begin{aligned} (D_\mathcal{E})^2 &= (D'_\mathcal{E})^2 - \frac{1}{2} \left[ \sum_{\alpha=1}^q (c(f_\alpha) c(\tau) + c(\tau) c(f_\alpha)) \nabla_{f_\alpha}^\mathcal{E} \right. \\ &\quad \left. - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha) c(\nabla_{f_\alpha} \tau) - \frac{1}{4} \|\tau\|^2 \right] \\ &= (D'_\mathcal{E})^2 + \nabla_\tau^\mathcal{E} - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha) c(\nabla_{f_\alpha} \tau) - \frac{1}{4} \|\tau\|^2. \end{aligned}$$

Taking into account (8), we get

$$(9) \quad \begin{aligned} (D_\mathcal{E})^2 &= \sum_{\alpha=1}^q (\nabla_{f_\alpha}^\mathcal{E})^* \nabla_{f_\alpha}^\mathcal{E} - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha) c(\nabla_{f_\alpha} \tau) - \frac{1}{4} \|\tau\|^2 \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) [R^\mathcal{E}(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)}^\mathcal{E}]. \end{aligned}$$

Finally, we use the formula

$$\frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) R^\mathcal{E}(f_\alpha, f_\beta) = \frac{K}{4} + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha) c(f_\beta) R^{\mathcal{E}/S}(f_\alpha, f_\beta),$$

that completes the proof of Theorem 4.

There is a natural action of  $Cl(Q_x)$  on  $\Lambda Q_x$  given by the formula

$$(10) \quad c(f) = \varepsilon_{f^*} - i_f, \quad f \in Q_x,$$

where  $\varepsilon_{f^*}$  denotes the exterior product by the covector  $f^* \in Q_x^*$  dual to  $f$ ,  $i_f$  the interior product by  $f$ .

Recall that the symbol map  $\sigma : Cl(Q_x) \rightarrow \Lambda Q_x$  is defined by

$$\sigma(a) = c(a)1, \quad a \in Cl(Q_x)$$

and the quantization map  $\mathbf{c} = \sigma^{-1} : \Lambda Q_x \rightarrow Cl(Q_x)$  is given by

$$\mathbf{c}(f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_k}) = c(f_{i_1}) c(f_{i_2}) \dots c(f_{i_k}),$$

where  $\{f_1, f_2, \dots, f_q\}$  is an orthonormal base in  $Q_x$ . These maps satisfy

$$(11) \quad \sigma(\mathbf{c}(v) \mathbf{c}(\omega)) = c(v) \omega, \quad v \in Q_x, \quad \omega \in \Lambda(Q_x).$$

So, for any  $\omega_1 \in \Lambda^i Q_x$  and  $\omega_2 \in \Lambda^j Q_x$  we have

$$\sigma(\mathbf{c}(\omega_1)\mathbf{c}(\omega_2)) = \omega_1 \wedge \omega_2 \mod \Lambda^{i+j-2} Q_x.$$

By (10) and (11), we have

$$\sigma\left(\sum_{\alpha=1}^q c(f_\alpha)c(\nabla_{f_\alpha}\tau)\right) = \sum_{\alpha=1}^q c(f_\alpha)\nabla_{f_\alpha}\tau = \sum_{\alpha=1}^q (\varepsilon_{f_\alpha}^* - i_{f_\alpha})\nabla_{f_\alpha}\tau.$$

Recall the following lemma.

**Lemma 6.** *Let  $f_1, f_2, \dots, f_q$  be a local orthonormal basis of  $T^H M$  and  $f_1^*, f_2^*, \dots, f_q^*$  be the dual basis of  $T^H M^*$ . Then on  $C^\infty(M, \Lambda T^H M^*)$  we have*

$$d_H = \sum_{\alpha=1}^q \varepsilon_{f_\alpha}^* \nabla_{f_\alpha}, \quad d_H^* = - \sum_{\alpha=1}^q i_{f_\alpha} \nabla_{f_\alpha} + i_\tau.$$

By Lemma 6, we have

$$\sigma\left(\sum_{\alpha=1}^q c(f_\alpha)c(\nabla_{f_\alpha}\tau)\right) = d_H\tau + d_H^*\tau - \|\tau\|^2.$$

Assume that the bundle-like metric  $g_M$  on  $M$  satisfies the assumption: the mean curvature form  $\tau$  is a basic one-form. As shown by Dominguez [5], such a bundle-like metric exists for any Riemannian foliation. Under this assumption, we have [14] (see also [20]):  $d\tau = 0$ . This fact implies that

$$\sigma\left(\sum_{\alpha=1}^q c(f_\alpha)c(\nabla_{f_\alpha}\tau)\right) = d_H^*\tau - \|\tau\|^2 \in C^\infty(M, \Lambda^0 T^* M) = C^\infty(M),$$

and, therefore,

$$\sum_{\alpha=1}^q c(f_\alpha)c(\nabla_{f_\alpha}\tau) = d_H^*\tau - \|\tau\|^2.$$

So we come to the following consequence (cf. [7]):

**Theorem 7.** *Assume that the bundle-like metric  $g_M$  on  $M$  satisfies the assumption: the mean curvature form  $\tau$  is a basic one-form. Then we have:*

$$\begin{aligned} (D_{\mathcal{E}})^2 &= \sum_{\alpha=1}^q (\nabla_{f_\alpha}^{\mathcal{E}})^* \nabla_{f_\alpha}^{\mathcal{E}} - \frac{1}{2} d_H^* \tau + \frac{1}{4} \|\tau\|^2 \\ &\quad + \frac{K}{4} + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha)c(f_\beta) [R^{\mathcal{E}/S}(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)}], \end{aligned}$$

where  $f_1, \dots, f_q$  is a local orthonormal basis of  $T^H M$ .

#### 4. TRANSVERSAL BOCHNER FORMULA

In this Section, we derive the Lichnerowicz formula for the transverse Laplacian on a compact manifold  $M$  equipped with a Riemannian foliation  $\mathcal{F}$ , which can be naturally called a Bochner formula.

**4.1. The transverse signature and Laplace operators.** Suppose that  $(M, \mathcal{F})$  is a compact Riemannian foliated manifold equipped with a bundle-like metric  $g_M$ . The decomposition (1) induces a bigrading on  $\Lambda T^*M$ :

$$\Lambda^k T^*M = \bigoplus_{i=0}^k \Lambda^{i, k-i} T^*M,$$

where

$$\Lambda^{i,j} T^*M = \Lambda^i T\mathcal{F}^* \otimes \Lambda^j T^H M^*.$$

In this bigrading, the de Rham differential  $d$  can be written as

$$d = d_F + d_H + \theta,$$

where  $d_F$  and  $d_H$  are first order differential operators (the tangential de Rham differential and the transversal de Rham differential accordingly), and  $\theta$  is a zero order differential operator.

The transverse signature operator is a first order differential operator in  $C^\infty(M, \Lambda T^H M^*)$  given by

$$D_H = d_H + d_H^*,$$

and the transversal Laplacian is a second order transversally elliptic differential operator in  $C^\infty(M, \Lambda T^H M^*)$  given by

$$\Delta_H = d_H d_H^* + d_H^* d_H.$$

**Theorem 8.** *Let  $f_1, \dots, f_q$  be a local orthonormal basis of  $T^H M$ . Then we have the following formula*

$$\Delta_H = \sum_{\alpha=1}^q \nabla_{f_\alpha}^* \nabla_{f_\alpha} + \sum_{\alpha=1}^q \varepsilon_{f_\alpha}^* i_{\nabla_{f_\alpha} \tau} - \sum_{\alpha, \beta} \varepsilon_{f_\alpha} i_{f_\beta} \left( R(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)} \right).$$

We give two proofs of Theorem 8. The first proof derives the theorem from Theorem 4, whereas the second proof is direct and makes no use of Theorem 4.

**4.2. The first proof.** Consider a transverse Clifford module  $\mathcal{E} = \Lambda T^H M^*$  which equipped with a natural leafwise flat Clifford connection and the corresponding transverse Dirac operator  $D_{\Lambda T^H M^*}$  acting in  $C^\infty(M, \Lambda T^H M^*)$ . The Clifford action of  $Cl(Q)$  on  $\mathcal{E}$  is defined by the formula (10). By Lemma 6 and (10), we have

$$(12) \quad D_{\Lambda T^H M^*} = d_H + d_H^* - \frac{1}{2}(\varepsilon_{\tau^*} + i_\tau).$$

By (12), it follows that

$$\begin{aligned} \Delta_H &= \left( D_{\Lambda T^H M^*} + \frac{1}{2}(\varepsilon_{\tau^*} + i_\tau) \right)^2 - d_H^2 - (d_H^*)^2 \\ &= D_{\Lambda T^H M^*}^2 - d_H^2 - (d_H^*)^2 + \frac{1}{2} \left( D_{\Lambda T^H M^*}(\varepsilon_{\tau^*} + i_\tau) + (\varepsilon_{\tau^*} + i_\tau) D_{\Lambda T^H M^*} \right) \end{aligned}$$

$$+ \frac{1}{4}(\varepsilon_{\tau^*} i_{\tau} + i_{\tau} \varepsilon_{\tau^*}).$$

By Theorem 4, it follows that

$$\begin{aligned} D_{\Lambda^{TH}M^*}^2 &= \sum_{\alpha=1}^q \nabla_{f_{\alpha}}^* \nabla_{f_{\alpha}} - \frac{1}{2} \sum_{\alpha=1}^q (\varepsilon_{f_{\alpha}} - i_{f_{\alpha}})(\varepsilon_{\nabla_{f_{\alpha}} \tau} - i_{\nabla_{f_{\alpha}} \tau}) - \frac{1}{4} \|\tau\|^2 \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} (\varepsilon_{f_{\alpha}} - i_{f_{\alpha}})(\varepsilon_{f_{\beta}} - i_{f_{\beta}})[R^{\Lambda^{TH}M^*}(f_{\alpha}, f_{\beta}) - \nabla_{\mathcal{R}(f_{\alpha}, f_{\beta})}]. \end{aligned}$$

As in the classical case, we have

$$(13) \quad \frac{1}{2} \sum_{\alpha, \beta} \varepsilon_{f_{\alpha}} \varepsilon_{f_{\beta}} R^{\Lambda^{TH}M^*}(f_{\alpha}, f_{\beta}) = 0,$$

$$(14) \quad \frac{1}{2} \sum_{\alpha, \beta} i_{f_{\alpha}} i_{f_{\beta}} R^{\Lambda^{TH}M^*}(f_{\alpha}, f_{\beta}) = 0.$$

The following lemma seems to be well known, but we didn't find an appropriate reference.

**Lemma 9.** *We have*

$$d_H^2 = -\frac{1}{2} \sum_{\alpha, \beta} \varepsilon_{f_{\alpha}} \varepsilon_{f_{\beta}} \nabla_{\mathcal{R}(f_{\alpha}, f_{\beta})}$$

and

$$(d_H^*)^2 = -\frac{1}{2} \sum_{\alpha, \beta} i_{f_{\alpha}} i_{f_{\beta}} \nabla_{\mathcal{R}(f_{\alpha}, f_{\beta})} - \sum_{\alpha} i_{f_{\alpha}} i_{\nabla_{f_{\alpha}} \tau}.$$

*Proof.* (1) By Lemma 6, we have

$$d_H^2 \omega = \sum_{\alpha, \beta} f_{\alpha} \wedge \nabla_{f_{\alpha}} f_{\beta} \wedge \nabla_{f_{\beta}} \omega + \sum_{\alpha, \beta} f_{\alpha} \wedge f_{\beta} \wedge \nabla_{f_{\alpha}} \nabla_{f_{\beta}} \omega.$$

As above, write  $\nabla_{f_{\alpha}} f_{\beta} = \sum_{\gamma} a_{\alpha\beta}^{\gamma} f_{\gamma}$ , where  $a_{\alpha\beta}^{\gamma} = -a_{\alpha\gamma}^{\beta}$ . Then, for the first term, we have

$$\begin{aligned} \sum_{\alpha, \beta} f_{\alpha} \wedge \nabla_{f_{\alpha}} f_{\beta} \wedge \nabla_{f_{\beta}} \omega &= \sum_{\alpha, \beta, \gamma} a_{\alpha\beta}^{\gamma} f_{\alpha} \wedge f_{\gamma} \wedge \nabla_{f_{\beta}} \omega \\ &= - \sum_{\alpha, \beta, \gamma} a_{\alpha\gamma}^{\beta} f_{\alpha} \wedge f_{\gamma} \wedge \nabla_{f_{\beta}} \omega \\ &= - \sum_{\alpha, \gamma} f_{\alpha} \wedge f_{\gamma} \wedge \nabla_{\nabla_{f_{\alpha}} f_{\gamma}} \omega \\ &= -\frac{1}{2} \sum_{\alpha, \gamma} f_{\alpha} \wedge f_{\gamma} \wedge (\nabla_{\nabla_{f_{\alpha}} f_{\gamma}} - \nabla_{\nabla_{f_{\gamma}} f_{\alpha}}) \omega \\ &= -\frac{1}{2} \sum_{\alpha, \gamma} f_{\alpha} \wedge f_{\gamma} \wedge (\nabla_{[f_{\alpha}, f_{\gamma}] + \mathcal{R}(f_{\alpha}, f_{\gamma})} \omega \end{aligned}$$

For the second term, we use the definition of the curvature  $R$  and (13):

$$\begin{aligned} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge \nabla_{f_\alpha} \nabla_{f_\beta} \omega &= \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge (\nabla_{f_\alpha} \nabla_{f_\beta} - \nabla_{f_\beta} \nabla_{f_\alpha}) \omega \\ &= \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge (\nabla_{[f_\alpha, f_\beta]} + R(f_\alpha, f_\beta)) \omega \\ &= \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge \nabla_{[f_\alpha, f_\beta]} \omega. \end{aligned}$$

(2) Similarly, using Lemma 6, we get

$$(d_H^*)^2 = \sum_{\alpha,\beta}^q i_{f_\alpha} i_{\nabla_{f_\alpha} f_\beta} \nabla_{f_\beta} + \sum_{\alpha,\beta}^q i_{f_\alpha} i_{f_\beta} \nabla_{f_\alpha} \nabla_{f_\beta} - \sum_{\alpha} (i_\tau i_{f_\alpha} \nabla_{f_\alpha} + i_{f_\alpha} \nabla_{f_\alpha} i_\tau).$$

Repeating the same arguments as above, we obtain

$$\sum_{\alpha,\beta}^q i_{f_\alpha} i_{\nabla_{f_\alpha} f_\beta} \nabla_{f_\beta} + \sum_{\alpha,\beta}^q i_{f_\alpha} i_{f_\beta} \nabla_{f_\alpha} \nabla_{f_\beta} = -\frac{1}{2} \sum_{\alpha,\beta} i_{f_\alpha} i_{f_\beta} \nabla_{\mathcal{R}(f_\alpha, f_\beta)}.$$

For the third term, we have

$$\begin{aligned} \sum_{\alpha} (i_\tau i_{f_\alpha} \nabla_{f_\alpha} + i_{f_\alpha} \nabla_{f_\alpha} i_\tau) &= \sum_{\alpha} ((i_\tau i_{f_\alpha} + i_{f_\alpha} i_\tau) \nabla_{f_\alpha} + i_{f_\alpha} i_{\nabla_{f_\alpha} \tau}) \\ &= \sum_{\alpha} i_{f_\alpha} i_{\nabla_{f_\alpha} \tau}. \end{aligned}$$

□

By (9) and Lemma 9, it follows that

$$\begin{aligned} D_{\Lambda T^H M^*}^2 - d_H^2 - (d_H^*)^2 &= \sum_{\alpha=1}^q \nabla_{f_\alpha}^* \nabla_{f_\alpha} - \frac{1}{2} \sum_{\alpha=1}^q (\varepsilon_{f_\alpha} - i_{f_\alpha}) (\varepsilon_{\nabla_{f_\alpha} \tau} - i_{\nabla_{f_\alpha} \tau}) - \frac{1}{4} \|\tau\|^2 \\ &\quad - \sum_{\alpha,\beta} \varepsilon_{f_\alpha} i_{f_\beta} [R^{\Lambda T^H M^*}(f_\alpha, f_\beta) - \nabla_{\mathcal{R}(f_\alpha, f_\beta)}] + \sum_{\alpha} i_{f_\alpha} i_{\nabla_{f_\alpha} \tau}. \end{aligned}$$

Recall that  $D_{\Lambda T^H M^*}$  is given by the formula

$$D_{\Lambda T^H M^*} = \sum_{\alpha=1}^q (\varepsilon_{f_\alpha} - i_{f_\alpha}) \left( \nabla_{f_\alpha}^{\Lambda T^H M^*} - \frac{1}{2} g_M(\tau, f_\alpha) \right)$$

Using the identity  $(\varepsilon_u - i_u)(\varepsilon_v + i_v) + (\varepsilon_v + i_v)(\varepsilon_u - i_u) = 0$  for any  $u$  and  $v$ , we get

$$D_{\Lambda T^H M^*}(\varepsilon_{\tau^*} + i_\tau) + (\varepsilon_{\tau^*} + i_\tau) D_{\Lambda T^H M^*}$$

$$\begin{aligned}
&= \sum_{\alpha=1}^q (\varepsilon_{f_\alpha} - i_{f_\alpha}) \left[ \nabla_{f_\alpha}^{\Lambda T^H M^*} - \frac{1}{2} g_M(\tau, f_\alpha), \varepsilon_{\tau^*} + i_\tau \right] \\
&= \sum_{\alpha=1}^q (\varepsilon_{f_\alpha} - i_{f_\alpha}) (\varepsilon_{\nabla_{f_\alpha} \tau^*} + i_{\nabla_{f_\alpha} \tau}).
\end{aligned}$$

The above identities and the formula  $\varepsilon_{\tau^*} i_\tau + i_\tau \varepsilon_{\tau^*} = \|\tau\|^2$  immediately complete the proof.

**4.3. A direct proof.** Here we indicate a direct proof of Theorem 8. By Lemma 6, we have

$$d_H^* d_H = - \sum_{\alpha, \beta} i_{f_\alpha} \varepsilon_{\nabla_{f_\alpha} f_\beta^*} \nabla_{f_\beta} - \sum_{\alpha, \beta} i_{f_\alpha} \varepsilon_{f_\beta^*} \nabla_{f_\alpha} \nabla_{f_\beta} + \sum_{\alpha=1}^q i_\tau \varepsilon_{f_\alpha^*} \nabla_{f_\alpha}$$

and

$$d_H d_H^* = - \sum_{\alpha, \beta} \varepsilon_{f_\alpha} i_{\nabla_{f_\alpha} f_\beta} \nabla_{f_\beta} - \sum_{\alpha, \beta} \varepsilon_{f_\alpha} i_{f_\beta} \nabla_{f_\alpha} \nabla_{f_\beta} + \sum_{\alpha=1}^q \varepsilon_{f_\alpha^*} \nabla_{f_\alpha} i_\tau.$$

From these identities, it follows that

$$\begin{aligned}
\Delta_H &= - \sum_{\alpha, \beta} (i_{f_\alpha} \varepsilon_{\nabla_{f_\alpha} f_\beta^*} + \varepsilon_{f_\alpha} i_{\nabla_{f_\alpha} f_\beta}) \nabla_{f_\beta} - \sum_{\alpha, \beta} (i_{f_\alpha} \varepsilon_{f_\beta^*} + \varepsilon_{f_\alpha} i_{f_\beta}) \nabla_{f_\alpha} \nabla_{f_\beta} \\
&\quad + \sum_{\alpha=1}^q (i_\tau \varepsilon_{f_\alpha^*} \nabla_{f_\alpha} + \varepsilon_{f_\alpha^*} \nabla_{f_\alpha} i_\tau).
\end{aligned}$$

Writing  $\nabla_{f_\alpha} f_\beta = \sum_\gamma a_{\alpha\beta}^\gamma f_\gamma$ , where  $a_{\alpha\beta}^\gamma = -a_{\alpha\gamma}^\beta$ , we get

$$\begin{aligned}
\sum_{\alpha, \beta} (i_{f_\alpha} \varepsilon_{\nabla_{f_\alpha} f_\beta^*} + \varepsilon_{f_\alpha} i_{\nabla_{f_\alpha} f_\beta}) \nabla_{f_\beta} &= \sum_{\alpha, \beta, \gamma} a_{\alpha\beta}^\gamma (i_{f_\alpha} \varepsilon_{f_\gamma} + \varepsilon_{f_\alpha} i_{f_\gamma}) \nabla_{f_\beta} \\
&= - \sum_{\alpha, \beta, \gamma} a_{\alpha\gamma}^\beta (i_{f_\alpha} \varepsilon_{f_\gamma} + \varepsilon_{f_\alpha} i_{f_\gamma}) \nabla_{f_\beta} \\
&= - \sum_{\alpha, \gamma} (i_{f_\alpha} \varepsilon_{f_\gamma} + \varepsilon_{f_\alpha} i_{f_\gamma}) \nabla_{\nabla_{f_\alpha} f_\gamma} \\
&= - \sum_{\alpha, \gamma} (i_{f_\alpha} \varepsilon_{f_\gamma} + \varepsilon_{f_\gamma} i_{f_\alpha}) \nabla_{\nabla_{f_\alpha} f_\gamma} \\
&\quad - \sum_{\alpha, \gamma} (\varepsilon_{f_\alpha} i_{f_\gamma} - \varepsilon_{f_\gamma} i_{f_\alpha}) \nabla_{\nabla_{f_\alpha} f_\gamma} \\
&= - \nabla_{\sum_\alpha \nabla_{f_\alpha} f_\alpha} - \sum_{\alpha, \gamma} \varepsilon_{f_\alpha} i_{f_\gamma} \nabla_{\nabla_{f_\alpha} f_\gamma - \nabla_{f_\gamma} f_\alpha} \\
&= - \nabla_{\sum_\alpha \nabla_{f_\alpha} f_\alpha} - \sum_{\alpha, \gamma} \varepsilon_{f_\alpha} i_{f_\gamma} \nabla_{[f_\alpha, f_\gamma] + \mathcal{R}(f_\alpha, f_\gamma)}.
\end{aligned}$$



We also have

$$\begin{aligned}
\sum_{\alpha, \beta} (i_{f_\alpha} \varepsilon_{f_\beta}^* + \varepsilon_{f_\alpha} i_{f_\beta}) \nabla_{f_\alpha} \nabla_{f_\beta} &= \sum_{\alpha, \beta} (i_{f_\alpha} \varepsilon_{f_\beta}^* + \varepsilon_{f_\beta} i_{f_\alpha}) \nabla_{f_\alpha} \nabla_{f_\beta} \\
&\quad + \sum_{\alpha, \beta} (\varepsilon_{f_\alpha} i_{f_\beta} - \varepsilon_{f_\beta} i_{f_\alpha}) \nabla_{f_\alpha} \nabla_{f_\beta} \\
&= \sum_{\alpha} (\nabla_{f_\alpha})^2 + \sum_{\alpha, \beta} \varepsilon_{f_\alpha} i_{f_\beta} (\nabla_{f_\alpha} \nabla_{f_\beta} - \nabla_{f_\beta} \nabla_{f_\alpha}) \\
&= \sum_{\alpha} (\nabla_{f_\alpha})^2 + \sum_{\alpha, \beta} \varepsilon_{f_\alpha} i_{f_\beta} (\nabla_{[f_\alpha, f_\beta]} + R(f_\alpha, f_\beta)).
\end{aligned}$$

Taking into account (7), we immediately complete the proof of Theorem 8.

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